ECE 346: Digital Signal Processing | Spring 2021
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Recitation #10: Introduction to the Discrete Fourier Transform

Objective & Outline

- Problems $1-5$: recitation problems
- Problem 6: self-assessment problem

The problems start on the following page.

Problem 1 (DFT Practice)**.** Recall that the N-point DFT "analysis" and inverse DFT "synthesis" equations are given by

$$X[K] = \sum_{n=0}^{N-1} x[n] W_N^{Kn}, \tag{1}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[K] W_N^{-Kn}, \tag{2}$$

where $W_N = e^{-j\frac{2\pi}{N}}$, respectively.

1. Consider the signal discrete-time signal $x[n]$:

$$x[n] = \delta[n] - 4\delta[n-2]. \tag{3}$$

   (a) Compute the 3-point DFT of $x[n]$, say $X_1[K]$.

   (b) Compute the 6-point DFT of $x[n]$, say $X_2[K]$.

   (c) What is the relationship between $X_1[K]$ and $X_2[2K]$ for $k = 0, 1, 2$?

2. Given the sequence

$$X[K] = 1 - 3W_{16}^{k} + 5W_{16}^{7k}, \tag{4}$$

   compute the 16-point inverse DFT of $X[K]$.

Solutions :-

1. We are given the signal

$$x[n] = \delta[n] - 4\delta[n-2].$$

(a) Explicitly using the DFT formula, the 3-point DFT of $x[n]$ is

$$X_1[k] = \sum_{n=0}^{2} x[n]\, W_N^{kn}$$

$$= x[0]\, W_N^{k\cdot 0} + x[1]\, W_N^{k\cdot 1} + x[2]\, W_N^{k\cdot 2}$$

$$= 1 - 4W_N^{2k} = 1 - 4W_3^{2k}$$

(b) Similarly, the 6-point DFT of $x[n]$ is

$$X_2[k] = 1 - 4W_6^{2k}.$$

Note that the only value that changed was 'N'.

(c) To identify the relationship between, $X_1[k]$ and $X_2[2k]$, we plug in

$$W_N = e^{-j\frac{2\pi}{N}}$$

into $X_1[k]$ and $X_2[k]$:

$$X_1[k] = 1 - 4W_3^{2k} \qquad\qquad X_2[k] = 1 - 4W^{2k}$$

$$= 1 - 4e^{-j\frac{2\pi}{3}\cdot 2k} \qquad\qquad = 1 - 4e^{-j\frac{2\pi}{6}\cdot 2k}$$

$$= 1 - 4e^{-j\frac{4\pi}{3}k} \qquad\qquad = 1 - 4e^{-j\frac{2\pi}{3}k}$$

Expanding upon $X_2[k]$, we can compute $X_2[2k]$

$$X_2[2k] = 1 - 4e^{-j\frac{2\pi}{3}\cdot 2k}$$

$$= 1 - 4e^{-j\frac{4\pi}{3}k}.$$

The relationship is that

$$X_1[k] = X_2[2k],$$

i.e., the 3-point DFT is equivalent to the 6-point DFT indexed by 2k for this sequence.

2. Recall the synthesis equation:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-kn} .$$

Expanding the 16-point DFT $X[k]$, we get

$$X[K] = x[0] + x[1] W_{16}^{k} + x[2] W_{16}^{2k} + \dots + x[n-1] W_N^{k(n-1)} .$$

Reading off these terms, we have

$$\begin{cases} x[0] = 1 \\ x[1] = -3 \\ x[7] = 5 \\ 0, \quad \text{otherwise.} \end{cases}$$

Thus,

$$x[n] = \delta[n] - 3\delta[n-1] + 5\delta[n-7].$$

$\blacksquare$

Problem 2 (DTFT & DFT)**.** Let us define the N-point DFT of *any* sequence as the samples of its DTFT at points $\omega_k = \frac{2\pi k}{N}$. Using this definition of the DFT, compute the 8-point DFT of the following sequences:

(a) $x_1[n] = \delta[n-1] + \delta[n-2] + 3\delta[n-5]$

(b) $x_2[n] = \left(\frac{1}{2}\right)^n u[n]$

(c) $x_3[n] = \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$

Solutions :-

In the previous problem, we computed the N-point DFT by explicitly using the formula. In this problem, we want to compute the DFT using the relation

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} .$$

(a) Taking the DTFT of the sequence

$$x_1[n] = \delta[n-1] + \delta[n-3] + 3\delta[n-5]$$

is simply

$$X_1(e^{j\omega}) = e^{-j\omega} + e^{-j3\omega} + 3e^{-j5\omega} .$$

Using the relationship $\omega = \frac{2\pi k}{N}$:

$$X_1[k] = X_1(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = e^{-j\frac{2\pi k}{8}} + e^{-j\frac{6\pi k}{8}} + 3e^{-j\frac{10\pi k}{8}}$$

Simplifying further, we get

$$X_1[k] = W_8^{k} + W_8^{3k} + 3W_8^{5k} .$$

(b) Recall that the DTFT formula is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} .$$

Plugging in $x_2[n]$:

$$X_2(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5\, e^{-j\omega})^n$$

$$= \frac{1}{1-0.5\, e^{-j\omega}} .$$

Using the relationship $\omega = \frac{2\pi k}{N}$:

$$X_2[k] = X_2(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = \frac{1}{1-0.5\, e^{-j\frac{\pi k}{4}}} , \quad \text{for } k = 0, 1, \ldots, 7 .$$

(c) Taking the DTFT of $x_3[n]$ yields

$$X_3(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi. \end{cases}$$

For $X_3[K]$, we need to plug in $\omega = \frac{2\pi k}{8}$ $(N=8)$ for $k = 0, 1, \ldots, 7$. This yields

$$\omega = \{ \underbrace{0}_{K=0}, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{3\pi}{2}, \underbrace{\frac{7\pi}{4}}_{K=7} \}.$$

Plugging in these ω values into $X_3(e^{j\omega})$, we obtain

$$X[k] = \{ \underbrace{1}_{K=0}, 1, 0, \ldots, 0, \underbrace{1}_{K=7} \}.$$

$\boxtimes$

Problem 3 (Computing DFT)**.** Let the discrete-time signal $x[n]$ be defined as

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

with $|a| < 1$.

(a) Provide a closed-form expression for the DTFT of $x[n]$.

(b) Provide a closed-form expression for the N-point DFT of $x[n]$ using the "analysis" equation.

(c) Could you have obtained the N-point DFT of $x[n]$ without having to explicitly use the DFT formula? Justify your answer.

Solutions :-

(a) Explicitly using the DTFT formula:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$
$$= \sum_{n=0}^{N-1} a^n e^{-j\omega n}$$
$$= \sum_{n=0}^{N-1} (a e^{-j\omega})^n$$

Using the formula

$$\sum_{i=0}^{n-1} a r^i = a\left(\frac{1-r^n}{1-r}\right) \quad \text{for } |r|<1,$$

the closed-form expression is

$$X(e^{j\omega}) = \frac{1-(ae^{-j\omega})^N}{1-ae^{-j\omega}}.$$

(b) Explicitly using the N-point DFT formula:

$$X[K] = \sum_{n=0}^{N-1} x[n]\, W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} a^n W_N^{Kn} = \sum_{n=0}^{N-1} (aW_N^K)^n$$

$$= \frac{1-(aW_N^K)^N}{1-aW_N^K}$$

$$= \frac{1-a^N e^{-j2\pi K}}{1-ae^{-j2\pi K/N}}.$$

(c) We could have used the relationship

$$X[K] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi K}{N}}$$

and have gotten (b) from (a).

$\boxtimes$

Problem 4 (Computing DFT)**.** Let the discrete-time signal $x[n]$ be defined as

$$x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

(a) Provide a closed-form expression for the DTFT of $x[n]$.

(b) Provide a closed-form expression for the N-point DFT of $x[n]$ using the "analysis" equation.

(c) Could you have obtained the N-point DFT of $x[n]$ without having to explicitly use the DFT formula? Justify your answer.

Solutions :-

(a) Explicitly using the DTFT formula:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$
$$= \sum_{n=0}^{N-1} e^{j\omega_0 n} \cdot e^{-j\omega n}$$
$$= \sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n}$$

Using the formula

$$\sum_{i=0}^{n-1} ar^i = a\left(\frac{1-r^n}{1-r}\right) \quad \text{for } |r|<1,$$

the closed-form expression is

$$X(e^{j\omega}) = \frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}}$$

(b) Explicitly using the N-point DFT formula:

$$X[K] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$
$$= \sum_{n=0}^{N-1} e^{j\omega_0 n} W_N^{kn} = \sum_{n=0}^{N-1} e^{j(\omega_0 - \frac{2\pi k}{N})n}$$
$$= \frac{1-e^{j(\omega_0-\frac{2\pi k}{N})N}}{1-e^{j(\omega_0-\frac{2\pi k}{N})}} .$$

(c) We could have used the relationship

$$X[K] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

and have gotten (b) from (a).

$\blacksquare$

Problem 5 (Computation Time of the DFT)**.** Suppose it takes a computer

$$C_0 N \log_2(N) \tag{7}$$

seconds to compute the N-point DFT of a sequence using an FFT algorithm. It takes a computer 0.5 seconds to compute a 1024-point DFT using the algorithm above. How long will it take the same computer to compute a 4096-point DFT of the same sequence using the same algorithm?

Solutions :-

We are given that the FFT algorithm takes

$$t = 0.5 = C_0 N \log_2(N) \text{ seconds}$$

for $N = 1024$. Solving for C_0, we get

$$0.5 = C_0 (1024) \log_2 (1024)$$

$$\Rightarrow \quad C_0 = \frac{0.5}{1024 \log_2 (1024)}$$

$$= 4.88 \times 10^{-5} .$$

Plugging in C_0 and $N = 4096$:

$$t = (4.88 \times 10^{-5})(4096) \log_2 (4096)$$

$$= 2.398 \text{ seconds.}$$

Problem 6 (Self-assessment)**.** Compute the N-point inverse DFT of the following sequence for a fixed ω_0:

$$X[K] = e^{j\omega_0 K}(u[K] - u[K-N]). \tag{8}$$

Can you express your final answer in terms of a ratio of two sine functions?

Solutions :-

Using the inverse - DFT formula,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\omega_0 k} W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(e^{j\omega_0} W_N^{-n} \right)^k$$

Using the formula previously used,

$$x[n] = \frac{1}{N} \left(\frac{1 - \left(e^{j\omega_0} W_N^{-n} \right)^N}{1 - e^{j\omega_0} W_N^{-n}} \right)$$

$$= \frac{1}{N} \left(\frac{1 - e^{j\omega_0 N} e^{-\frac{j2\pi n N}{N}}}{1 - e^{j\omega_0} W_N^{-n}} \right)$$

$$= \frac{1}{N} \left(\frac{1 - e^{j\omega_0 N}}{1 - e^{j\omega_0} W_N^{-n}} \right)$$

Taking out a factor from the numerator and denominator:

$$x[n] = \frac{1}{N} \cdot \frac{e^{j\omega_0 N/2}}{e^{j\omega_0/2} W_N^{-n/2}} \left[\frac{e^{-j\omega_0 N/2} - e^{j\omega_0 N/2}}{e^{-j\omega_0/2} W_N^{n/2} - e^{j\omega_0/2} W_N^{-n/2}} \right]$$

$$= \frac{1}{N} e^{j\left(\omega_0 \frac{(N-1)}{2} - \frac{\pi n}{N} \right)} \cdot \left[\frac{\sin\left(\omega_0 N/2 \right)}{\sin\left(\omega_0/2 + \pi n/N \right)} \right].$$